Advanced Studies in Contemporary Mathematics
34 (2024), No. 4, pp. 293 - 306

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.4.293

Some new modular equations of composite degrees

Shruthi C. Bhat, B. R. Srivatsa Kumar and Taekyun Kim*

Department of Mathematics, Manipal Institute of Technology,
Manipal Academy of Higher Education, Manipal, India

*Department of Mathematics, Kwangwoon University,
Seoul 139701, Republic of Korea

E-mail:kwangwoonmath@hanmail.net, tkkim@kw.ac.kr

Abstract

S. Ramanujan recorded many modular equations in his notebooks. In our study, we obtain few new mixed modular equations by using some familiar modular equations which are analogous to Ramanujan's modular equations.

Keywords: Modular equations, theta functions, q-identities.
2020 Mathematics Subject Classification: Primary: 11B65, 11F27, Secondary: 33D15.

1 Introduction

Ramanujan [12, vol II] in his notebooks has documented some modular equations. Further, these are proved by B. C. Berndt [8] using parametrization and modular forms. Also C. Adiga et al. [2], M. S. M. Naika [9, 10], M. S. M. Naika and S. Chandankumar [11] and N. Saikia and J. Chetry [13] have also obtained some modular equations of signature three. Ramanujan [12] begins his study on modular equations in Chapter 15 by defining

$$F(x) := (1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} x^n = {}_1F_0\left(\frac{1}{2}; x\right), \qquad |x| < 1.$$

He then states a trivial identity

$$F\left(\frac{2t}{1+t}\right) = (1+t)F(t^2). \tag{1}$$

After setting $\alpha = 2t/(1+t)$ and $\beta = 2t^2$ in (1), Ramanujan offers a modular equation of degree two given by

$$\beta(2-\alpha)^2 = \alpha^2,$$

and the factor $(1+t)$ in (1) is called the multiplier. Further Ramanujan has developed theory of elliptic functions in which q is replaced by one or the other functions q^n, for $n = 3, 4$ and 6.

$$q_n := q_n(x) := exp\left(-\pi csc(\pi/r) \frac{{}_2F_1(\frac{1}{n}, \frac{n-1}{n}; 1; 1-x)}{{}_2F_1(\frac{1}{n}, \frac{n-1}{n}; 1; x)}\right) \qquad 0 < x < 1,$$

where ${}_2F_1$ represent the classical hypergeometric function defined as follows:

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} z^m, \qquad |z| < 1$$

and

$$(\alpha)_m = \alpha(\alpha+1)\ldots(\alpha+m-1).$$

These theories are now known as the theory of signature n, where $n = 3, 4$ and 6. For $n = 3$ and 4, the theories are known as cubic and quartic theories respectively. Let us now take up a modular equation as given in the literature. A modular equation of degree n in the theory of signature two, is an equation relating α and β that is induced by

$$n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}.$$

Then, we say that β is of degree n over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)$.

This paper is classified as follows. In Section 2, we list some P-Q type theta function identities which will be utilized to demonstrate modular equations of signature two. Further in Section 3, we prove composite degrees of modular equations. Before that we state some basic results, which are useful in our main results. All through this paper, we assume $|q| < 1$. The standard q-shifted factorial is defined as

$$(a;q)_0 = 1, \qquad (a;q)_n := \prod_{i=1}^{n}(1-aq^{i-1}) \qquad \text{and} \qquad (a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i).$$

In classical theory, Ramanujan [1, 12] has defined theta function [6, p. 36] as follows:

$$\begin{aligned}
\varphi(q) &:= \mathfrak{f}(q,q) = 1 + \sum_{i=1}^{\infty} q^{i^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} = \Theta_3(0,q),\\
\psi(q) &:= \mathfrak{f}(q,q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},\\
f(-q) &:= \mathfrak{f}(-q,-q^2) = \sum_{i=0}^{\infty}(-1)^i q^{i(3i-1)/2} + \sum_{i=1}^{\infty}(-1)^i q^{i(3i+1)/2} = (q;q)_\infty,
\end{aligned}$$

where Θ_3 is the classical theta function. We note that $f(-q^n) = f_n$ throughout the sequel.

2 Preliminaries

Here, we list certain modular equations as preliminaries which would be useful for our main results.

Lemma 2.1. *[7, p.209, Entry 55] Let*

$$P = \frac{f_1^2}{q^{1/2}f_7^2} \quad \textit{and} \quad Q = \frac{f_2^2}{qf_{14}^2}.$$

Then

$$PQ+\frac{49}{PQ}=\left(\frac{Q}{P}\right)^3-8\frac{Q}{P}-8\frac{P}{Q}+\left(\frac{P}{Q}\right)^3.$$

Lemma 2.2. *[7, p. 236, Entry 68] Let*

$$P=\frac{f_1}{q^{1/4}f_7}\quad and\quad Q=\frac{f_3}{q^{3/4}f_{21}}.$$

Then

$$PQ+\frac{7}{PQ}=\left(\frac{Q}{P}\right)^2-3+\left(\frac{P}{Q}\right)^2.$$

Lemma 2.3. *[7, p. 236, Entry 71] Let*

$$P=\frac{f_1}{q^{1/4}f_7}\quad and\quad Q=\frac{f_5}{q^{5/4}f_{35}}.$$

Then

$$PQ-5+\frac{49}{PQ}=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}\right)^2-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3.$$

Lemma 2.4. *[7, p. 210, Entry 56] Let*

$$P=\frac{f_1}{q^{1/3}f_9}\quad and\quad Q=\frac{f_2}{q^{2/3}f_{18}}.$$

Then

$$P^3+Q^3=P^2Q^2+3PQ.$$

Lemma 2.5. *[7, p. 211, Entry 57] Let*

$$P=\frac{f_1}{q^{1/2}f_{13}}\quad and\quad Q=\frac{f_2}{qf_{26}}.$$

Then

$$PQ+\frac{13}{PQ}=\left(\frac{Q}{P}\right)^3-4\frac{Q}{P}-4\frac{P}{Q}+\left(\frac{P}{Q}\right)^3.$$

Lemma 2.6. *[7, p. 237, Entry 72] Let*

$$P=\frac{f_1}{q^{1/2}f_{13}}\quad and\quad Q=\frac{f_3}{q^{3/2}f_{39}}.$$

Then

$$PQ+\frac{13}{PQ}=\left(\frac{Q}{P}\right)^2-3\frac{Q}{P}-3-3\frac{P}{Q}+\left(\frac{P}{Q}\right)^2.$$

Lemma 2.7. *Let*

$$P=\frac{\varphi(-q)}{q^{1/4}\psi(q^2)}\qquad and\qquad Q_n=\frac{\varphi(-q^n)}{q^{n/4}\psi(q^{2n})}.$$

Then

$$PQ_2+\frac{16}{PQ_2}-\left(\frac{P}{Q_2}\right)^2-\left(\frac{Q_2}{P}\right)^2-6=0. \tag{2}$$

and

$$\left(\frac{P}{Q_5}\right)^3+\left(\frac{Q_5}{P}\right)^3+10\left[\left(\frac{P}{Q_5}\right)^2+\left(\frac{Q_5}{P}\right)^2\right]+15\left[\frac{P}{Q_5}+\frac{Q_5}{P}\right]-20$$
$$=(PQ_5)^2+\frac{256}{(PQ_5)^2}. \tag{3}$$

For the proof of (2) and (3), one may refer [17].

3 Main results

In this section, we state and prove certain modular equations of composite degrees with the help of known modular equations listed in the previous section.

Theorem 3.1. *If β,γ,δ,η and ζ have degrees 2, 7, 14, 4 and 28 over α respectively and if*

$$r=\frac{z_1z_{14}}{z_2z_7}\left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/12} \quad \textit{and} \quad s=\frac{z_2z_{28}}{z_4z_{14}}\left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/12},$$

then we have

$$(as^2-br^4)(br^2-as^4)=49r^4s^4(1-r^2s^2)^2,$$

where

$$a=r^6-8r^4-8r^2+1 \qquad \textit{and} \qquad b=s^6-8s^4-8s^2+1.$$

Proof. Let

$$L=\frac{f_1^2}{q^{1/2}f_7^2},\quad M=\frac{f_2^2}{qf_{14}^2}\quad \text{and}\quad N=\frac{f_4^2}{q^2f_{28}^2}. \tag{4}$$

On employing (4) in Lemma 2.1, we obtain

$$LM+\frac{49}{LM}=\left(\frac{M}{L}\right)^3-8\frac{M}{L}-8\frac{L}{M}+\left(\frac{L}{M}\right)^3 \tag{5}$$

and

$$MN+\frac{49}{MN}=\left(\frac{N}{N}\right)^3-8\frac{N}{M}-8\frac{M}{N}+\left(\frac{M}{N}\right)^3. \tag{6}$$

From [6, p. 124, Entry 11(ii)], we have

$$f(-q)=\sqrt{z}2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}. \tag{7}$$

Using (7) in (5) and (6), we obtain

$$\frac{L}{M}=\frac{z_1z_{14}}{z_2z_7}\left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/12}=r \tag{8}$$

and

$$\frac{M}{N}=\frac{z_2z_{28}}{z_4z_{14}}\left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/12}=s. \tag{9}$$

Employing (8) and (9) in (5) and (6) respectively, we deduce

$$r^4M^4 - aM^2 + 49r^2 = 0 \tag{10}$$

and

$$s^2M^4 - bM^2 + 49s^4 = 0, \tag{11}$$

where a and b are as defined as in Theorem 3.1. From (10) and (11), we deduce

$$\frac{M^4}{49(br^2 - as^4)} = \frac{M^2}{49r^2s^2(1 - r^2s^2)} = \frac{1}{as^2 - br^4},$$

which implies

$$M^4 = \frac{49(br^2 - as^4)}{as^2 - br^4} \tag{12}$$

and

$$M^2 = \frac{49r^2s^2(1 - r^2s^2)}{as^2 - br^4}. \tag{13}$$

On combining (12) and (13), we obtain the desired result. □

Theorem 3.2. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees 3, 7, 21, 9 and 63 over α respectively and if*

$$r = \sqrt{\frac{z_1 z_{21}}{z_3 z_7}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4} \right)^{1/24} \quad \text{and} \quad s = \sqrt{\frac{z_3 z_{63}}{z_9 z_{21}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4} \right)^{1/24},$$

then we have

$$(br - as^3)(br^3 - as) = 7rs(1 - r^2s^2)^2,$$

where

$$a = r^4 - 3r^2 + 1 \qquad \text{and} \qquad b = s^4 - 3s^2 + 1.$$

Proof. Let

$$L = \frac{f_1}{q^{1/4} f_7}, \quad M = \frac{f_3}{q^{3/4} f_{21}} \quad \text{and} \quad N = \frac{f_9}{q^{9/4} f_{63}}. \tag{14}$$

On employing (14) in Lemma 2.2, we obtain

$$LM + \frac{7}{LM} = \left(\frac{M}{L}\right)^2 - 3 + \left(\frac{L}{M}\right)^2 \tag{15}$$

and

$$MN + \frac{7}{MN} = \left(\frac{N}{M}\right)^2 - 3 + \left(\frac{M}{N}\right)^2. \tag{16}$$

Using (7) in (15) and (16), we obtain

$$\frac{L}{M} = \sqrt{\frac{z_1 z_{21}}{z_3 z_7}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4} \right)^{1/24} = r \tag{17}$$

and

$$\frac{M}{N} = \sqrt{\frac{z_3 z_{63}}{z_9 z_{21}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4} \right)^{1/24} = s. \tag{18}$$

Employing (17) and (18) in (15) and (16) respectively, we deduce

$$r^3M^4 - aM^2 + 7r = 0 \tag{19}$$

and

$$sM^4 - bM^2 + 7s^3 = 0, \tag{20}$$

where a and b are as defined as in Theorem 3.2. From (19) and (20), we deduce

$$\frac{M^4}{7(br - as^3)} = \frac{M^2}{7rs(1 - r^2s^2)} = \frac{1}{br^3 - as},$$

which implies

$$M^4 = \frac{7(br - as^3)}{br^3 - as} \tag{21}$$

and

$$M^2 = \frac{7rs(1 - r^2s^2)}{br^3 - as}. \tag{22}$$

On combining (21) and (22), we obtain the desired result. □

Theorem 3.3. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees 5, 7, 35, 25 and 175 over α respectively and if*

$$r = \sqrt{\frac{z_1 z_{35}}{z_5 z_7}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} \quad \text{and} \quad s = \sqrt{\frac{z_5 z_{175}}{z_{25} z_{35}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24},$$

then we have

$$(br - as^5)(as - br^5) = 49r^2s^2(1 - r^4s^4)^2,$$

where

$$a = -r^6 - 5r^5 + 5r^3 - 5r + 1 \qquad \text{and} \qquad b = -s^6 - 5s^5 + 5s^3 - 5s + 1.$$

Proof. Let

$$L = \frac{f_1}{q^{1/4} f_7}, \quad M = \frac{f_5}{q^{5/4} f_{35}} \quad \text{and} \quad N = \frac{f_{25}}{q^{25/4} f_{175}}. \tag{23}$$

On employing (23) in Lemma 2.3, we obtain

$$(LM)^2 - 5 + \frac{49}{(LM)^2} = \left(\frac{M}{L}\right)^3 - 5\left(\frac{M}{L}\right)^2 - 5\left(\frac{L}{M}\right)^2 - \left(\frac{L}{M}\right)^3 \tag{24}$$

and

$$(MN)^2 - 5 + \frac{49}{(MN)^2} = \left(\frac{N}{M}\right)^3 - 5\left(\frac{N}{M}\right)^2 - 5\left(\frac{M}{N}\right)^2 - \left(\frac{M}{N}\right)^3. \tag{25}$$

Using (7) in (24) and (25), we obtain

$$\frac{L}{M} = \sqrt{\frac{z_1 z_{35}}{z_5 z_7}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} = r \tag{26}$$

and

$$\frac{M}{N} = \sqrt{\frac{z_5 z_{175}}{z_{25} z_{35}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24} = s. \tag{27}$$

Employing (26) and (27) in (24) and (25) respectively, we deduce

$$r^5M^8 - aM^4 + 49r = 0 \tag{28}$$

and

$$sM^8 - bM^4 + 49s^5 = 0, \tag{29}$$

where a and b are as defined as in Theorem 3.3. From (28) and (29), we deduce

$$\frac{M^8}{49(br - as^5)} = \frac{M^4}{49rs(1 - r^4s^4)} = \frac{1}{as - br^5},$$

which implies

$$M^8 = \frac{49(br - as^5)}{as - br^5} \tag{30}$$

and

$$M^4 = \frac{49rs(1 - r^4s^4)}{as - br^5}. \tag{31}$$

On combining (30) and (31), we obtain the desired result. □

Theorem 3.4. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees 2, 9, 18, 4 and 36 over α respectively and if*

$$r = \sqrt{\frac{z_1 z_{18}}{z_2 z_9}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4} \right)^{1/24} \quad \textit{and} \quad s = \sqrt{\frac{z_2 z_{36}}{z_4 z_{18}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4} \right)^{1/24},$$

then we have

$$(br - as)(as^2 - br^2) = 3r^2s^2(s - r)^2,$$

where

$$a = r^3 + 1 \qquad \textit{and} \qquad b = s^3 + 1.$$

Proof. Let

$$L = \frac{f_1}{q^{1/3} f_9}, \quad M = \frac{f_2}{q^{2/3} f_{18}} \quad \text{and} \quad N = \frac{f_4}{q^{4/3} f_{36}}. \tag{32}$$

On employing (32) in Lemma 2.4, we obtain

$$L^3 + M^3 = L^2M^2 + 3LM \tag{33}$$

and

$$M^3 + N^3 = M^2N^2 + 3MN. \tag{34}$$

Using (7) in (33) and (34), we obtain

$$\frac{L}{M} = \sqrt{\frac{z_1 z_{18}}{z_2 z_9}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4} \right)^{1/24} = r \tag{35}$$

and

$$\frac{M}{N} = \sqrt{\frac{z_2 z_{36}}{z_4 z_{18}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4} \right)^{1/24} = s. \tag{36}$$

Employing (35) and (36) in (33) and (34) respectively, we deduce

$$r^2M^2 - aM + 3r = 0 \tag{37}$$

and

$$s^2M^2 - bM + 3s = 0, \tag{38}$$

where a and b are as defined as in Theorem 3.4. From (37) and (38), we deduce

$$\frac{M^2}{3(br-as)} = \frac{M}{3rs(s-r)} = \frac{1}{as^2-br^2},$$

which implies

$$M^2 = \frac{3(br-as)}{as^2-br^2} \tag{39}$$

and

$$M = \frac{3rs(s-r)}{as^2-br^2}. \tag{40}$$

On combining (39) and (40), we obtain the desired result. □

Theorem 3.5. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees 2, 13, 26, 4 and 52 over α respectively and if*

$$r = \sqrt{\frac{z_1z_{26}}{z_2z_{13}}}\left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} \quad \textit{and} \quad s = \sqrt{\frac{z_2z_{52}}{z_4z_{26}}}\left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24},$$

then we have

$$(as^2-br^4)(br^2-as^4) = 13r^4s^4(1-r^2s^2)^2,$$

where

$$a = r^6 - 4r^4 - 4r^2 + 1 \qquad \textit{and} \qquad b = s^6 - 4s^4 - 4s^2 + 1.$$

Proof. Let

$$L = \frac{f_1}{q^{1/2}f_{13}}, \quad M = \frac{f_2}{qf_{26}} \quad \text{and} \quad N = \frac{f_4}{q^2f_{52}}. \tag{41}$$

On employing (41) in Lemma 2.5, we obtain

$$LM + \frac{13}{LM} = \left(\frac{M}{L}\right)^3 - 4\frac{M}{L} - 4\frac{L}{M} + \left(\frac{L}{M}\right)^3 \tag{42}$$

and

$$MN + \frac{13}{MN} = \left(\frac{N}{M}\right)^3 - 4\frac{N}{M} - 4\frac{M}{N} + \left(\frac{M}{N}\right)^3. \tag{43}$$

Using (7) in (42) and (43), we obtain

$$\frac{L}{M} = \sqrt{\frac{z_1z_{26}}{z_2z_{13}}}\left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} = r \tag{44}$$

and

$$\frac{M}{N} = \sqrt{\frac{z_2z_{52}}{z_4z_{26}}}\left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24} = s. \tag{45}$$

Employing (44) and (45) in (42) and (43) respectively, we deduce

$$r^4M^4 - aM^2 + 13r^2 = 0 \tag{46}$$

and

$$s^2M^4 - bM^2 + 13s^4 = 0, \tag{47}$$

where a and b are as defined as in Theorem 3.5. From (46) and (47), we deduce

$$\frac{M^4}{13(br^2 - as^4)} = \frac{M^2}{13r^2s^2(1 - r^2s^2)} = \frac{1}{as^2 - br^4},$$

which implies

$$M^4 = \frac{13(br^2 - as^4)}{as^2 - br^4} \tag{48}$$

and

$$M^2 = \frac{13r^2s^2(1 - r^2s^2)}{as^2 - br^4}. \tag{49}$$

On combining (48) and (49), we obtain the desired result. □

Theorem 3.6. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees 3, 13, 39, 9 and 117 over α respectively and if*

$$r = \sqrt{\frac{z_1 z_{39}}{z_3 z_{13}}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} \quad \text{and} \quad s = \sqrt{\frac{z_3 z_{117}}{z_9 z_{39}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24},$$

then we have

$$(as - br^3)(br - as^3) = 13r^2s^2(1 - r^2s^2)^2,$$

where

$$a = r^4 - 3r^3 - 3r^2 - 3r + 1 \qquad \text{and} \qquad b = s^4 - 3s^3 - 3s^2 - 3s + 1.$$

Proof. Let

$$L = \frac{f_1}{q^{1/2} f_{13}}, \quad M = \frac{f_3}{q^{3/2} f_{39}} \quad \text{and} \quad N = \frac{f_9}{q^{9/2} f_{117}}. \tag{50}$$

On employing (50) in Lemma 2.6, we obtain

$$LM + \frac{13}{LM} = \left(\frac{M}{L}\right)^2 - 3\frac{M}{L} - 3 - 3\frac{L}{M} + \left(\frac{L}{M}\right)^2 \tag{51}$$

and

$$MN + \frac{13}{MN} = \left(\frac{N}{M}\right)^3 - 3\frac{N}{M} - 3 - 3\frac{M}{N} + \left(\frac{M}{N}\right)^2. \tag{52}$$

Using (7) in (51) and (52), we obtain

$$\frac{L}{M} = \sqrt{\frac{z_1 z_{39}}{z_3 z_{13}}} \left(\frac{\alpha\delta(1-\alpha)^4(1-\delta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\right)^{1/24} = r \tag{53}$$

and

$$\frac{M}{N} = \sqrt{\frac{z_3 z_{117}}{z_9 z_{39}}} \left(\frac{\beta\eta(1-\beta)^4(1-\eta)^4}{\delta\zeta(1-\delta)^4(1-\zeta)^4}\right)^{1/24} = s. \tag{54}$$

Employing (53) and (54) in (51) and (52) respectively, we deduce

$$r^3M^4 - aM^2 + 13r = 0 \tag{55}$$

and

$$sM^4 - bM^2 + 13s^3 = 0, \tag{56}$$

where a and b are as defined as in Theorem 3.6. From (55) and (56), we deduce

$$\frac{M^4}{13(br - as^3)} = \frac{M^2}{13rs(1 - r^2s^2)} = \frac{1}{as - br^3},$$

which implies

$$M^4 = \frac{13(br - as^3)}{as - br^3} \tag{57}$$

and

$$M^2 = \frac{13rs(1 - r^2s^2)}{as - br^3}. \tag{58}$$

On combining (57) and (58), we obtain the desired result. □

Theorem 3.7. *If β and γ have degrees 3 and 9 over α respectively and if*

$$r = 4\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} \qquad \text{and} \qquad s = 4\left(\frac{(1-\beta)(1-\gamma)}{\beta\gamma}\right)^{1/4},$$

then

$$rs(br^3 - as^3)(ar - bs) = (r^4 - s^4)^2,$$

where

$$a = r^2 - 6r + 6 \qquad \text{and} \qquad b = s^2 - 6s + 6. \tag{59}$$

Proof. Let

$$L = \frac{\varphi(-q)}{q^{1/4}\psi(q^2)}, \qquad M = \frac{\varphi(-q^3)}{q^{3/4}\psi(q^6)} \qquad \text{and} \qquad N = \frac{\varphi(-q^9)}{q^{9/4}\psi(q^{18})}. \tag{60}$$

On employing (60) in (2), we have

$$LM + \frac{16}{LM} - \left(\frac{L}{M}\right)^2 - \left(\frac{M}{L}\right)^2 - 6 = 0 \tag{61}$$

and

$$MN + \frac{16}{MN} - \left(\frac{M}{N}\right)^2 - \left(\frac{N}{M}\right)^2 - 6 = 0. \tag{62}$$

From [6, pp. 122-123], we have

$$\varphi(-q) = \sqrt{z}(1-x)^{1/4} \tag{63}$$

and

$$\psi(q^2) = \frac{\sqrt{z}}{2}\left(\frac{x}{q}\right)^{1/4}. \tag{64}$$

From (63) and (64), we obtain

$$\frac{\varphi(-q)}{q^{1/4}\psi(q^2)} = 2\left(\frac{1-\alpha}{\alpha}\right)^{1/4}. \tag{65}$$

Using (65) in (60), we deduce

$$LM = 4\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} = r \tag{66}$$

and

$$MN = 4\left(\frac{(1-\beta)(1-\gamma)}{\beta\gamma}\right)^{1/4} = s \tag{67}$$

Using (66) and (67) in (61) and (62) respectively, we deduce

$$M^8 - arM^4 + r^4 = 0 \tag{68}$$

and

$$M^8 - bsM^4 + s^4 = 0, \tag{69}$$

where a and b are as defined as in Theorem 3.7. From (68) and (69), we deduce that

$$\frac{M^8}{rs(br^3 - as^3)} = \frac{M^4}{r^4 - s^4} = \frac{1}{ar - bs},$$

which implies

$$M^8 = \frac{rs(br^3 - as^3)}{ar - bs} \tag{70}$$

and

$$M^4 = \frac{r^4 - s^4}{ar - bs} \tag{71}$$

On combining (70) and (71), we complete the proof. □

Theorem 3.8. *If β and γ have degrees 5 and 25 over α respectively and if*

$$r = \left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{1/4} \qquad \text{and} \qquad s = \left(\frac{\gamma(1-\beta)}{\beta(1-\gamma)}\right)^{1/4}$$

then, we have

$$r^2s^2(a - br^2s^2)(b - ar^2s^2) = 256(1 - r^4s^4)^2,$$

where

$$a = u^3 + 10u^2 + 12u - 40 \qquad \text{and} \qquad b = v^3 + 10v^2 + 12v - 40,$$

with

$$u = r + \frac{1}{r} \qquad \text{and} \qquad v = s + \frac{1}{s}.$$

Proof. Let

$$L = \frac{\varphi(-q)}{q^{1/4}\psi(q^2)}, \qquad M = \frac{\varphi(-q^5)}{q^{5/4}\psi(q^{10})} \qquad \text{and} \qquad N = \frac{\varphi(-q^{25})}{q^{25/4}\psi(q^{50})}. \tag{72}$$

From (3), we have

$$\left(\frac{L}{M}\right)^3 + \left(\frac{M}{L}\right)^3 + 10\left[\left(\frac{L}{M}\right)^2 + \left(\frac{M}{L}\right)^2\right] + 15\left[\frac{L}{M} + \frac{M}{L}\right] - 20$$
$$= (LM)^2 + \frac{256}{(LM)^2} \tag{73}$$

and

$$\left(\frac{M}{N}\right)^3+\left(\frac{N}{M}\right)^3+10\left[\left(\frac{M}{N}\right)^2+\left(\frac{N}{M}\right)^2\right]+15\left[\frac{M}{N}+\frac{N}{M}\right]-20$$
$$=(MN)^2+\frac{256}{(MN)^2}. \tag{74}$$

Using (65) in (72), we observe that

$$\frac{L}{M}=\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{1/4}=r \tag{75}$$

and

$$\frac{M}{N}=\left(\frac{\gamma(1-\beta)}{\beta(1-\gamma)}\right)^{1/4}=s. \tag{76}$$

Using (75) and (76) in (73) and (74) respectively, we obtain

$$r^4M^8-ar^2M^4+256=0 \tag{77}$$

and

$$M^8-bs^2M^4+256s^4=0, \tag{78}$$

where a and b are as defined as in Theorem 3.8. From (77) and (78), we deduce that

$$\frac{M^8}{256s^2(b-ar^2s^2)}=\frac{M^4}{256(r^4s^4-1)}=\frac{1}{r^2(a-br^2s^2)},$$

which implies

$$M^8=\frac{256s^2(b-ar^2s^2)}{r^2(a-br^2s^2)} \tag{79}$$

and

$$M^4=\frac{256(r^4s^4-1)}{r^2(a-br^2s^2)}. \tag{80}$$

On combining (79) and (80), we complete the proof. □

4 Conclusions

Inspired by the earlier works that are cited herein, in the present article certain modular equations that are analogous to Ramanujan's identities are established. The authors believe that many of the recent works (see, for example, [3–5,14–16]) are potentially useful for indicating directions for further research on q-series, q-polynomials and q-differences based upon the subject matter which is related to that of our present investigation.

Acknowledgment

The first author would like to thank DST-INSPIRE, Department of Science and Technology, Government of India, India for providing the INSPIRE Fellowship (DST/INSPIRE/03/2022/004970) under which this work has been carried out.

References

[1] C. Adiga, B. C. Berndt, S. Bhargava and G. N. Watson, Chapter 16 of Ramanujan's second notebook: Theta functions and q-series, *Mem. Amer. Math. Soc.*, 315 (1985), 1–91.

[2] C. Adiga, T. Kim and M. S. M. Naika, Modular equations in the theory of signature 3 and P-Q identities, *Adv. Stud. Contemp. Math.* **7** (2003), 33–40.

[3] C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, On Ramanujan's cubic continued fraction and explicit evaluations of theta-functions, *Indian J. Pure Appl. Math.*, **35** (2004), 1047–1062.

[4] C. Adiga, N. A. S. Bulkhali, D. Ranganatha and H. M. Srivastava, Some new modular relations for the Rogers-Ramanujan type functions of order eleven with applications to partitions, *J. Number Theory*, **158** (2016), 281–297.

[5] D. Anuradha and B. R. Srivatsa Kumar, Some identities on modular equations of degree 5, *Publ. Inst. Math.*, **111** (2022), 101–110.

[6] B. C. Berndt, *Ramanujan's Notebooks: Part III*, Springer, New York (1991).

[7] B. C. Berndt, *Ramanujan's Notebooks: Part IV*, Springer, New York (1994).

[8] B. C. Berndt, *Ramanujan's Notebooks: Part V*, Springer, New York (1998).

[9] M. S. M. Naika, P-Q eta-function identities and computation of Ramanujan- Weber class invariants, *Journal of the Indian Mathematical Society*, **70** (1-4) (2003), 121–134.

[10] M. S. M. Naika, A note on cubic modular equations of degree two, *Tamsui Oxford J. Math. Sci.*, **22** (1) (2006), 1–8.

[11] M. S. M. Naika and S. Chandankumar, Some new Schläfli type modular equation of signature 4, *Ramanujan Math. Soc. Lect. Notes*, **14** (2010), 185–199.

[12] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay (1957).

[13] N. Saikia and J. Chetry, Some new modular equations in Ramanujan's alternate theory of signature 3, *The Ramanujan Journal*, **50** (2019) 163–194.

[14] H. M. Srivastava, M. P. Chaudhary and F. K. Wakene, A family of theta-function identities based upon q-binomial theorem and Heine's transformations, *Montes Taurus J. Pure Appl. Math.*, **2** (2020), 1–6.

[15] B. R. Srivatsa Kumar and Shruthi, New modular equations of signature three in the spirit of Ramanujan, *Filomat*, **34** (2020), 2847–2868.

[16] B. R. Srivatsa Kumar, A. K. Rathie, N. V. U. Sayinath and Shruthi, A note on modular equations of signature 2 and their evaluations, *Commun. Korean Math. Soc.*, **37** (2022), 31–44.

[17] K. R. Vasuki and B. R. Srivatsa Kumar, Certain identities for Ramanujan-Göllnitz-Gordon continued fraction, *Journal of computational and applied mathematics*, **187** (2006), 87-95.